# Applications of Short Rational Functions
# to Solving Integer Programming Problems

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# Getting started...

HOW MANY WAYS are there?

| | | | | | |
|---|---|---|---|---|---|
| ? | ? | ? | ? | ? | 338106 |
| ? | ? | ? | ? | ? | 574203 |
| ? | ? | ? | ? | ? | 678876 |
| ? | ? | ? | ? | ? | 1213008 |

Column sums (reading diagonally): $20202$, $142746$, $410755$, $100773$, $1222717$

# One more puzzle....

What is the optimal value and an optimal solution for the following problem?

Maximize

$$213x_1 - 1928x_2 - 11111x_3 - 2345x_4 + 9123x_5 - 12834x_6 - 123x_7 + 122331x_8$$

subject to

$$11948x_1 + 23330x_2 + 30635x_3 + 44197x_4 + 92754x_5 + 123389x_6 + 136951x_7 + 140745x_8 = 14215207,$$

$x_i \in \mathbb{Z}_+$ for $i = 1, 2, \ldots, 8$.

**Note:** Branch-and-bound method (CPLEX v.6.6) failed to solve this problem.

# How to solve

Let $P = \{x \in \mathbb{R}^d | Ax = a,\ Bx \leq b\}$, where $A$, $B$ are integral matrices and $a$, $b$ are integral vectors.

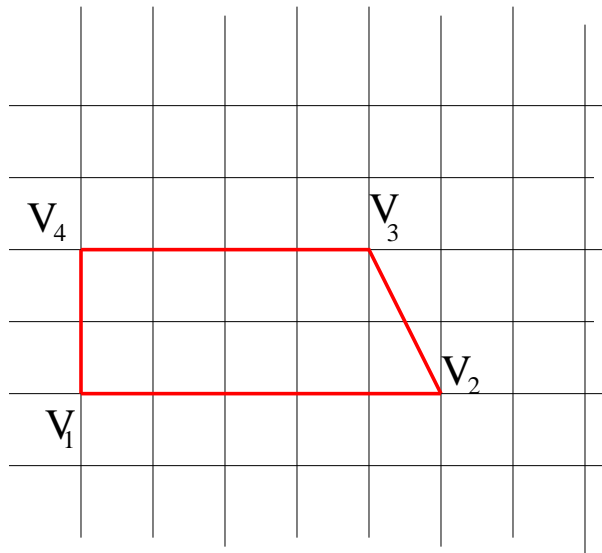**Tool**: The multivariate generating function

$$f(P, z) = \sum_{\alpha \in P \cap \mathbb{Z}^d} z^{\alpha},$$

where $z^{\alpha} = z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_d^{\alpha_d}$.

This is an infinite formal power series if $P$ is not bounded, but if $P$ is a polytope it is a polynomial.

# Example for $f(P, z)$

Let $V_1 = (0, 0)$, $V_2 = (5, 0)$, $V_3 = (4, 2)$, and $V_4 = (0, 2)$.

Each vertex is represented by the following monomials:

For $V_1 = (0,0)$, $z^{V_1} = z_1^0 z_2^0 = 1$.

For $V_2 = (5,0)$, $z^{V_2} = z_1^5 z_2^0 = z_1^5$.

For $V_3 = (4,2)$, $z^{V_3} = z_1^4 z_2^2$.

For $V_4 = (0,2)$, $z^{V_4} = z_1^0 z_2^2 = z_2^2$.

In this manner, we have $f(P, z)$ as the following:

$f(P, z) = z_1{}^5 + z_1{}^4 z_2 + z_1{}^4 + z_1{}^4 z_2{}^2 + z_2 z_1{}^3 + z_1{}^3 + z_1{}^3 z_2{}^2 + z_2 z_1{}^2 + z_1{}^2 + z_1{}^2 z_2{}^2 + z_1 z_2 + z_1 + z_1 z_2{}^2 + z_2{}^2 + z_2 + 1$.

# However...

The multivariate generating function $f(P, z)$ has exponentially many monomials even in fixed the dimension.

**Question**: How can we encode $f(P, z)$ in polynomial size if we fix the dimension?

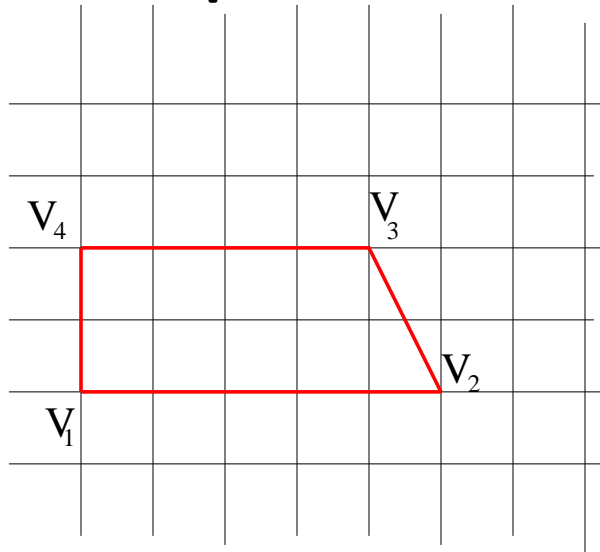**Answer**: We can encode $f(P, z)$ as a short sum of rational functions.

**Theorem**: [Barvinok (1993)]

Assume that we fix the dimension $d$ and suppose we have a rational convex polyhedron $P = \{\, u \in \mathbb{R}^d \; : \; Au \leq b \,\}$, where $A \in \mathbb{Z}^{m \times d}$ and $b \in \mathbb{Z}^m$. Then there exists a polynomial time algorithm to compute $f(P, z)$ in the following form of:

$$f(P, z) \quad = \quad \sum_{i \in I} \pm \frac{z^{u_i}}{(1 - z^{v_{1i}})(1 - z^{v_{2i}}) \ldots (1 - z^{v_{di}})}$$

where $u_i,\, v_{1i},\, \ldots v_{di} \in \mathbb{Z}^d$ for all $i \in I$.

# From the previous example



$$f(P,z) = z_1{}^5 + z_1{}^4 z_2 + z_1{}^4 + z_1{}^4 z_2{}^2 + z_2 z_1{}^3 + z_1{}^3 + z_1{}^3 z_2{}^2 + z_2 z_1{}^2 + z_1{}^2 + z_1{}^2 z_2{}^2 + z_1 z_2 + z_1 + z_1 z_2{}^2 + z_2{}^2 + z_2 + 1$$

$$= \frac{1}{(1-z_1)(1-z_2)} + \frac{z_1{}^5}{(1-z_1{}^{-1})(1-z_2)} + \frac{z_1^2}{(1-z_1)(1-z_2^{-1})} + \frac{z_1^5}{(1-z_1^{-1}z_2)(1-z_2^{-1})} +$$
$$\frac{z_1{}^4 z_2{}^2}{(1-z_2{}^{-1})(1-z_1)} - \frac{z_1^4 z_2^2}{(1-z_1^{-1}z_2{}^2)(1-z_1^{-1})}.$$

# LattE

- **INPUT**: Integral matrices $A$, $B$ and integral vectors $a$, $b$ for the polytope $P$.

- **OUTPUT**: $f(P, z)$ written as <span style="color:red">a short sum of rational functions</span>.

- **APPLICATIONS**:

  (A) Counting Problem,

  (B) Integer Programming,

  (C) Integer Feasibility Problem,

  (D) Computing a universal test set of a given integral matrix $A$.

# Answer to the puzzle

| | | | | | |
|---|---|---|---|---|---|
| ? | ? | ? | ? | ? | 338106 |
| ? | ? | ? | ? | ? | 574203 |
| ? | ? | ? | ? | ? | 678876 |
| ? | ? | ? | ? | ? | 1213008 |

2 0 2 0 2
1 4 2 7 4 6
4 1 0 7 5 5
1 0 0 7 7 3
1 2 2 7 1 7

31605282093011690945982204905214978774800496305802997262397.

# Answer to the puzzle

Maximize

$$213x_1 - 1928x_2 - 11111x_3 - 2345x_4 + 9123x_5 - 12834x_6 - 123x_7 + 122331x_8$$

subject to

$$11948x_1 + 23330x_2 + 30635x_3 + 44197x_4 + 92754x_5 + 123389x_6 + 136951x_7 + 140745x_8 = 14,215,207,$$

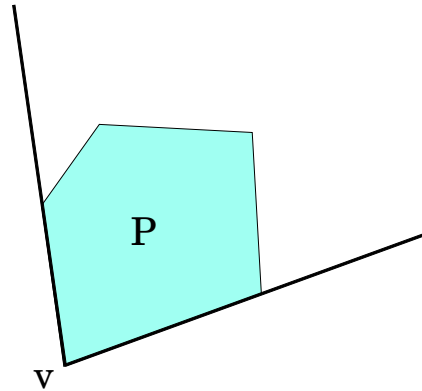$$x_i \in \mathbb{Z}_+ \text{ for } i = 1, 2, \ldots, 8.$$

The optimal value is $3471390$.
An optimal solution is $(853, 2, 0, 4, 0, 0, 0, 27)$.
The number of feasible integer points is $2047107$.
CPU Time: about 12 sec.

# Theory behind the implementation



**Theorem**[Brion, Lawrence]

Let $P$ be a convex polyhedron and let $V(P)$ be the vertex set of $P$. Let $K_v$ be the tangent cone at $v \in V(P)$. Then
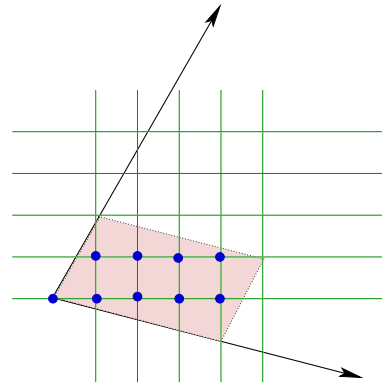
$$f(P, z) = \sum_{v \in V(P)} f(K_v, z).$$

# If we have a simple cone...

For a simple cone $K \subset \mathbb{R}^d$,

$$f(K, z) = \frac{\sum_{u \in \Pi \cap \mathbb{Z}^d} z^u}{(1 - z^{c_1})(1 - z^{c_2}) \ldots (1 - z^{c_d})}$$
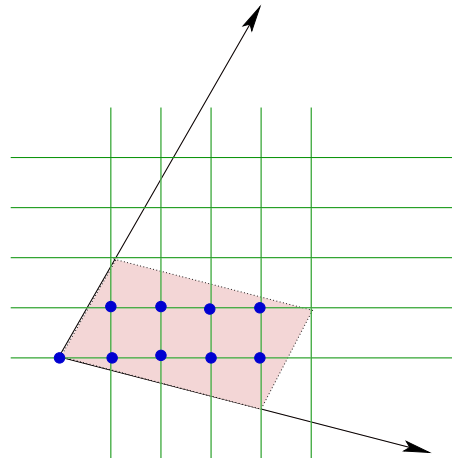
where $\Pi$ is the half open parallelepiped spanned by the rays of the cone $K$, $c_1, \ldots, c_d \in \mathbb{Z}^d$.
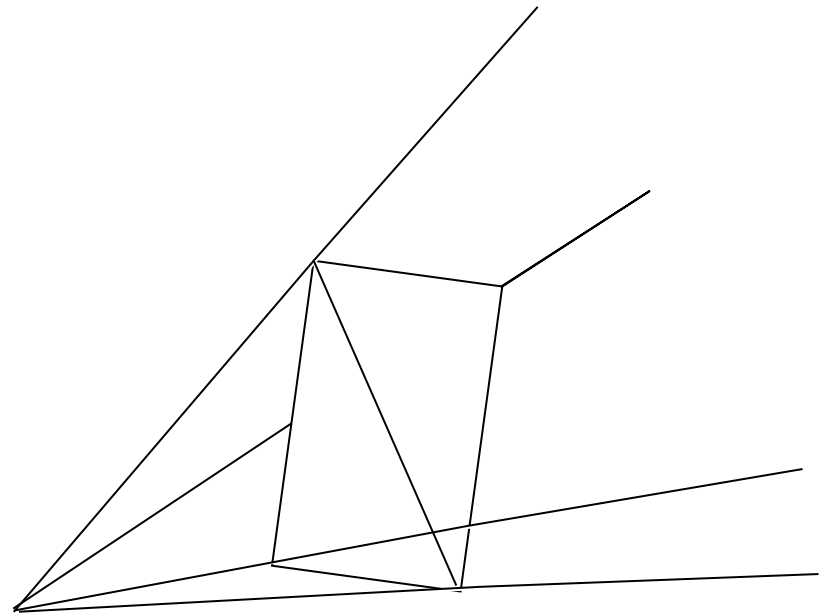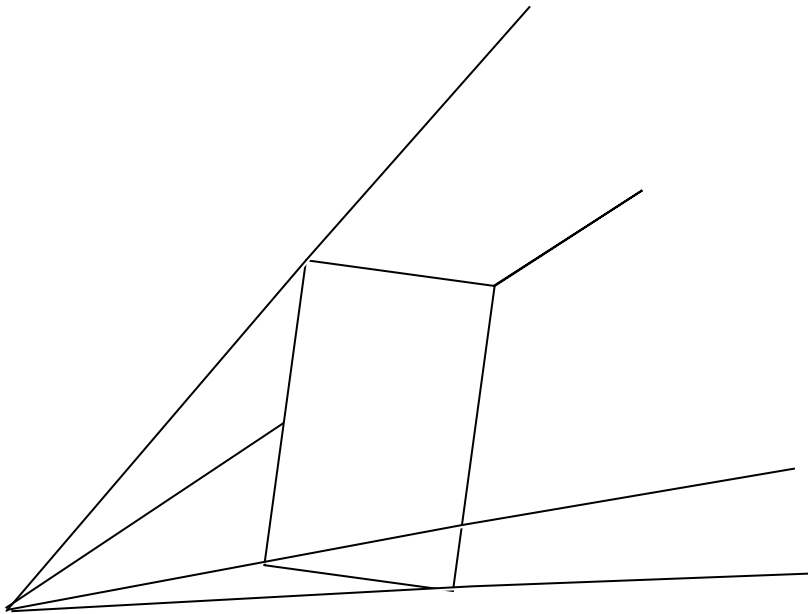
# Example

In this case, we have $d = 2$ and $c_1 = (1, 2)$, $c_2 = (4, -1)$. We have:

$$f(K, z) = \frac{z_1^4 z_2 + z_1^3 z_2 + z_1^2 z_2 + z_1 z_2 + z_1^4 + z_1^3 + z_1^2 + z_1 + 1}{(1 - z_1 z_2^2)(1 - z_1^4 z_2^{-1})}.$$

# If a cone $K$ is not simple....

We triangulate the cone into simple cones.

**Fact**: If the parallelepiped $\Pi$ for a simple cone $K$ has only one integral point $u$, then
$$f(K, z) = \frac{z^u}{\prod_{i=1}^{d}(1 - z^{c_i})}.$$

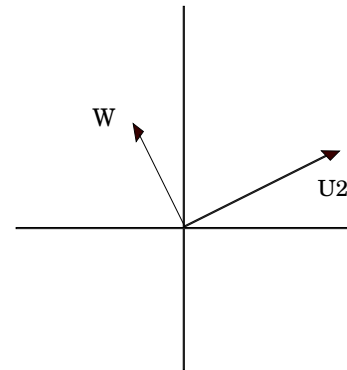**Goal**: Want to decompose $K$ into simple cones whose parallelepipeds $\Pi$ have ONLY one integral point.
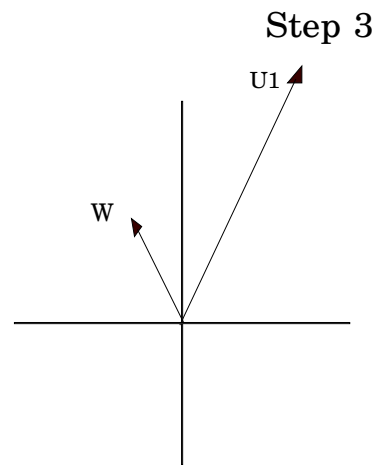
**Definition**: A *unimodular cone $K$* is a simple cone such that the half open parallelepiped generated by the rays of $K$ contains only one lattice point

# Barvinok's cone decomposition

**Theorem** [Barvinok] Fix the dimension $d$. Then there exists a polynomial time algorithm which decomposes a rational polyhedral cone $K \subset \mathbb{R}^d$ into unimodular cones $K_i$ with numbers $\epsilon_i \in \{-1, 1\}$ such that

$$f(K, z) = \sum_{i \in I} \epsilon_i f(K_i, z), \quad |I| < \infty.$$

# We do this process until all cones become unimodular.

Step 1

$a = (\det(U1 \mid U2))^{\frac{-1}{d}}$

U1

aU1

U2

aU2

Step 2

aU1

W

aU2

Step 3

U1

W

W

U2
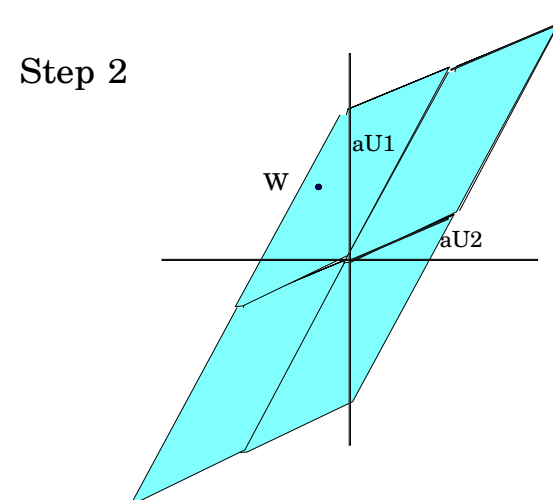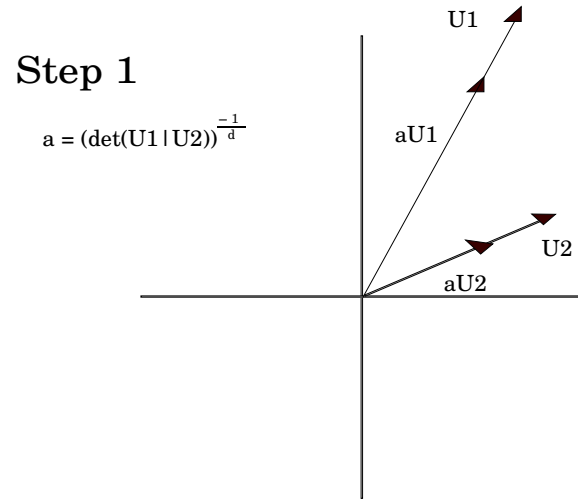
# We have so far...

1. Find the vertices of the given polytope and their defining tangent cones.

2. Triangulate and apply Barvinok's cone decomposition to each of the cones.

3. Obtain the signed rational function of each cone and sum them up.

We need to include lower dimensional cones and also apply the inclusion-exclusion principle, to get the multivariate generating function $f(P, z)$.

# Brion's polarization trick

**Lemma**: Let $K \subset \mathbb{R}^d$ be a cone. If $K$ contains a straight line then $f(K, z) \equiv 0$.

By this lemma we can avoid using the inclusion-and-exclusion principle and save time and memory. The trick is the following:

1. Polarize the tangent cone.

2. Decompose the polar into unimodular cones.

3. Polarize back each of the unimodular cones.

If we do this process, the multivariate function $f(K, z)$ for each lower dimensional cone $K$ becomes zero.

# Example

$d = 2$ and $K = \text{cone}\{(1, 0), (1, k)\}$ for some large $k \in \mathbb{Z}$.

The dual cone $K^* = \text{cone}\{(-k, 1), (0, -1)\}$.

Applying Barvinok's cone decomposition, we have

$$f(K^*, z) = f(K_1^*, z) + f(K_2^*, z) + f(K_3^*, z),$$

where $K_1^* = \text{cone}\{(-k, 1), (-1, 0)\}$, $K_2^* = \text{cone}\{(0, -1), (-1, 0)\}$, and $K_3^* = \text{cone}\{(-1, 0)\}$.

Applying Brion's trick, we have

$$f(K, z) = f(K_1, z) + f(K_2, z),$$

where $K_1 = \text{cone}\{(0, -1), (1, k)\}$ and $K_2 = \text{cone}\{(0, 1), (1, 0)\}$.

# Outline of the implementation

1. Find the vertices of the given polytope and their defining tangent cones.

2. Compute the polar cone to each of the cones.

3. Triangulate and apply Barvinok's cone decomposition to each of the polar cones.

4. Polarize back each of the full dimensional unimodular cones.

5. Obtain the signed rational function of each cone and sum them up.

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# Integer Programming

# via Short Rational Functions

# Integer programming

**Problem**: Given $A \in \mathbb{Z}^{m \times d}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^d$,

$$\max \ c \cdot x \ \text{ subject to } Ax \leq b, \ x \in \mathbb{Z}^d.$$

There are several algorithms to solve the integer programming problem using short rational functions, such as:

1. Barvinok's binary search (BBS) algorithm

2. The digging algorithm

3. The test-set (Gröbner bases, Graver bases, etc) algorithm

Barvinok's binary search (BBS) algorithm and the digging algorithm are implemented in `LattE`.

# Digging algorithm

**Problem**: Find the optimal value $M \in \mathbb{Z}$ and an optimal solution $x^* \in Z^d$ for $\max\{c \cdot x : x \in P \cap \mathbb{Z}^d\}$, where $P$ is a rational convex polyhedron.

If we make the substitutions $\Phi : \mathbb{C}^d \to \mathbb{C}$ such that $\Phi(z_i) = t^{c_i}$, then we will have that $\Phi(z^\alpha) = t^{c \cdot \alpha}$,

and we will thereby obtain

$$\Phi(f(P, z)) = \sum_{\alpha \in P \cap \mathbb{Z}^d} t^{c \cdot \alpha}$$

$$= k t^M + (\text{lower degree terms}).$$

**Note**: With the substitutions $z_i \to y_i t^{c_i}$ instead, we can obtain an optimal solution as well.

Suppose we have a sum of short rational functions for $P$:

$$f(P, z) \quad = \quad \sum_{i \in I} \pm \frac{z^{u_i}}{(1 - z^{v_{1i}})(1 - z^{v_{2i}}) \dots (1 - z^{v_{di}})}.$$

With the monomial substitutions $z_i \rightarrow y_i t^{c_i}$, we get:

$$g(P; y, t) = \sum_{i \in I} \pm \frac{y^{u_i} t^{c \cdot u_i}}{\prod_{j=1}^{d}(1 - y^{v_{ij}} t^{c \cdot v_{ij}})}.$$

We can assume that $c \cdot v_{ij} < 0$. If $c \cdot v_{ij} > 0$, then we use the following identity:

$$\frac{y^v}{(1 - y^u)} = -\frac{y^{v-u}}{(1 - y^{-u})}.$$

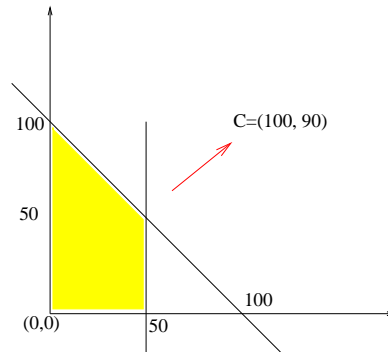We send $t$ to $\infty$ in the rational function

$$g(P; y, t) = \sum_{i \in I} \pm \frac{y^{u_i} t^{c \cdot u_i}}{\prod_{j=1}^d (1 - y^{v_{ij}} t^{c \cdot v_{ij}})}.$$

and we obtain the following power series:

$$ky^{x^*} t^M + (\text{terms whose degree in } t \text{ is } < M).$$

# Example

maximize $100x + 90y$ subject to $x + y \leq 100,\ x \leq 50,\ x, y \geq 0,\ x, y \in \mathbb{Z}^d.$



The rational generating function is:

$$\frac{1}{(1-x_1)(1-x_2)} + \frac{x_1^{50}}{(1-x_1^{-1})(1-x_2)} + \frac{x_2^{100}}{(1-x_2^{-1})(1-x_1 x_2^{-1})} + \frac{x_1^{50} x_2^{50}}{(1-x_2^{-1})(1-x_1^{-1} x_2)}.$$

# Example cont.

We apply the monomial substitution and apply the appropriate algebraic identities and we get:

$$\frac{t^{-190}}{(1-t^{-100})(1-t^{-90})} - \frac{t^{4910}}{(1-t^{-100})(1-t^{-90})} - \frac{t^{8990}}{(1-t^{-90})(1-t^{-10})} + \frac{t^{9500}}{(1-t^{-90})(1-t^{-10})}.$$

Sending $t$ to $\infty$, then this rational function approaches the power series

$$t^{9500} + (\text{lower degree terms}).$$

Therefore, the optimal value for this problem is $9500$.

# IP via Gröbner basis

First we define a term order from a vector $c$ as follows:

For all $\alpha, \beta \in \mathbb{Z}_+^d$, $\alpha \prec_c \beta$ if

- $c \cdot \alpha < c \cdot \beta$ or

- $c\alpha = c\beta$ and $\alpha \prec_{lex} \beta$.

For example, suppose $c = (1, 0, 2)$ and if we have $(3, 2, 7)$ and $(3, 5, 2)$ in $\mathbb{R}^3$, then we have $(1, 0, 2) \cdot (3, 2, 7) = 17$ and $(1, 0, 2) \cdot (3, 5, 2) = 13$. So, since $(1, 0, 2) \cdot (3, 5, 2) < (1, 0, 2) \cdot (3, 2, 7)$, we have $(3, 5, 2) \prec_c (3, 2, 7)$ and $x_1^3 x_2^5 x_3^2 \prec_c x_1^3 x_2^2 x_3^7$.

# What is a Gröbner basis??

Let $P = \{x \in \mathbb{R}^d : Ax = b, x \geq 0\} \neq \emptyset$, where $A \in \mathbb{Z}^{n \times d}$ and $b \in \mathbb{Z}^n$. Let $M$ be a finite set such that $M \subset \{x \in \mathbb{Z}^d : Ax = 0\}$ and let $\prec$ be any term order on $\mathbb{N}^d$. Then we define the graph $G_b$ such that:

- Nodes of $G_b$ are lattice points inside $P$.

- Draw a directed edge between a node $u$ and a node $v$ if and only if $u \prec v$ for $u - v \in M$.

If $G_b$ is acyclic and has a unique sink for all $b$ with $P \neq \emptyset$, then $M$ is a Gröbner basis for a toric ideal associate with a matrix $A$ with respect to $\prec$.

# Example (universal Gröbner basis)

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}.$$

Encode binomials $x^u - x^v$ in $n$ variables as $x^u y^v$ in $2n$ variables $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots, x_n^{\alpha_n}$ and $y^\beta = y_1^{\beta_1} y_2^{\beta_2}, \ldots, y_n^\beta$.

The output is a sum of $538$ simple rational functions such as $\left(1 - \frac{x_3 y_4}{x_1 y_2}\right)\left(1 - \frac{x_1 x_4 y_2}{x_3}\right)\left(1 - x_1 y_1\right)\left(1 - x_1 x_3 y_2{}^2\right)\left(1 - x_3 y_3\right)\left(1 - x_2 y_2\right).$

The number of binomials is 207188785065 and CPU time is 12 seconds.

# Algorithm

Let $P := \{x \in \mathbb{R}^d : Ax = b, \, x \geq 0\}$.

**Input:** $c \in \mathbb{Z}^d$, $A \in \mathbb{Z}^{n \times d}$, $b \in \mathbb{Z}^n$ and a feasible solution $v_0 \in P \cap \mathbb{Z}^d$.

**Output:** An optimal solution and the optimal value of minimize $c \cdot x$ subject to $x \in P \cap \mathbb{Z}^d$.

**Step 1:** Compute the Gröbner basis of $I_A$, the toric ideal associated with $A$ with the term order $\prec_c$.

**Step 2:** Compute the normal form $x^u$ of $x^{v_0}$ and return $u$ and $c \cdot u$, which are an optimal solution and the optimal value, respectively.

**Theorem** [De Loera, Haws, Hemmecke, Huggins, Sturmfels, Y. (04)]

Let $A \in \mathbb{Z}^{m \times d}$. Assuming that $m$, $d$ are fixed, there is a polynomial time algorithm to compute a short rational function $G(z)$ which represents the reduced Gröbner basis of the toric ideal $I_A$ w.r.t. any given term order $\prec$. Given $G$ and any monomial $x^a$, the following tasks can be performed in polynomial time:

1. Decide whether $x^a$ is in normal form with respect to $G(z)$.

2. Compute the normal form of $x^a$ modulo the Gröbner basis $G(z)$.

3. Let $b \in \mathbb{Z}^m$ and $c \in \mathbb{Z}^d$. Given a polyhedron $P = \{x | Ax = b,\ x \geq 0\}$, compute the integer programming problem:

$$\text{maximize } c \cdot x \text{ subject to } x \in P,\ x \in \mathbb{Z}^d.$$

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# Questions??

# Software

LattE is available from our website:

`http://www.math.ucdavis.edu/~latte`.

If you have any questions about LattE, please send me email at

ruriko@math.duke.edu.